Selected topics in quantum mecahnics

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Problem 1 Rabi Oscillations and Spin Echo

(a): Consider a two-state system that is perturbed with a frequency $\omega \neq \omega_0$. The Hamiltonian is given by ($\hbar = 1$)

$$H(t) = \begin{pmatrix} \omega_0/2 & g\cos(\omega t) \\ g\cos(\omega t) & -\omega_0/2 \end{pmatrix}.$$

Pass into a rotating basis by performing unitary time dependent transformation

$$U(t) = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}.$$

Assuming that the perturnation is weak ($g \ll \omega$) show the transformed Hamilton can be approximated by

$$\widetilde{H}(t) = \frac{1}{2}\begin{pmatrix} \Delta & g \\ g & -\Delta \end{pmatrix}.$$

Here $\Delta = \omega_0 - \omega$ is a detuning magnitude. Describe the evolution of the system governed by Hamiltonian $\widetilde{H}$ on the Bloch sphere. If the system at time $t = 0$ is in the ground state. Can it be driven into an excited stated by a perturbation $\widetilde{H}$?

(b): The Hamilton discussed in part (a) above describes a nuclear spin in a time independent magnetic field $B_z = \omega_0/g_I$ in z-direction in the presence of weak oscillating magnetic field $B_x = g/g_I$ in x-direction. In practice, instead of a single nuclear spin, one often deals with an ensemble of nuclear spins. Since no magnetic field is perfectly homogeneous, each nuclear spin will have a different detuning. Consider the following experiment.

At the beginning the system is in the ground state $|\psi(0)\rangle$. During the driving by the time dependent magnetic field the the detuning Δ can be neglected. At the first stage of the experiment the magnetic field in the x direction is turned on for a time t_1. This pulse of magnetic field results in the a phase shift $\pi/2$. As a results the system goes from the ground state $|\psi(0)\rangle = |g\rangle$ to a superposition $|\psi(t_1)\rangle = (|g\rangle - i\,|e\rangle)/\sqrt{2}$.

Find, how t_1 depends on g.

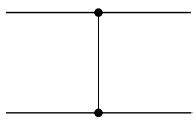
After that, the perturbation is switched off, for time interval t_2. Describe the change of a position of nuclear spins on the Bloch sphere during this interval.

Through the third driving interval $2t_1$ a π-pulse is applied. Afterwards that the perturbation is turned off for a time t_2. What happens with the spins on a Bloch sphere?

After a further $\pi/2$-pulls the system is in $|\psi(4t_1 + 2t_2)\rangle = |g\rangle$. This protocol is used in the magnetic resonance imaging. Explains, why the phases acquired during the three pulses need to be equal?

Problem 2. C-Phase-Gate.

We have seen in the lecture that C-NOT-Gate is a universal 2-qubit gate. Often, is it is more practical to build C-Phase-Gate instead. If the control qubit is in the state $|1\rangle$, the target qubit is transformed by σ_z. Otherwise there is no transformation. The symbol of thus gate is shown in the figure below.



Note, that there is no way to determine from the figure where the control Qbit is. Explain why?

To realize this gate one may use a system of two spins, described by the Hamiltonian

$$H(t) = \alpha(t)\sigma_z^{(1)} + \beta(t)\sigma_z^{(2)} + \gamma(t)\sigma_z^{(1)}\sigma_z^{(2)}\,.$$

Each spin is placed in a magnetic field, proportional to $\alpha(t)$ and $\beta(t)$ correspondingly. In addition there is Ising type interaction between the spins, controlled by $\gamma(t)$.

We need to find how an experimentalist should choose the functions $\alpha(t)$, $\beta(t)$ and $\gamma(t)$, to make C-phase gate.

Write down the Hamiltonian and evolution operator $U(t) = \exp[-i \int_0^t dt'\, H(t')]$ in the matrix form and determine the condition for $A = \int_0^t dt'\, \alpha(t')$, etc. Note, that the global phase can be arbitrary.